A brief history of (p, q) theorems

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Postdoc Colloquium

I. The convex case

A family of sets $\mathcal{S}$ is *n-consistent* if any subfamily of at most n sets in $\mathcal{S}$ has nonempty intersection.

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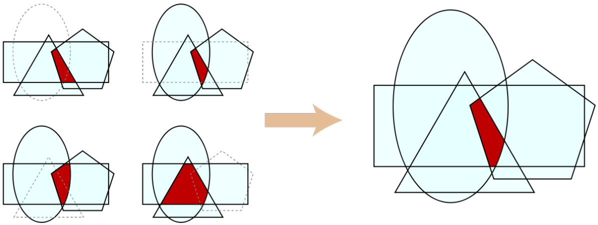
Helly's theorem (1913)

Let $\mathcal{S}$ be a finite family of convex subsets of $\mathbb{R}^d$. If $\mathcal{S}$ is $(d+1)$-consistent then $\cap\mathcal{S} \neq \emptyset$.



By a simple compactness argument this is also true for infinite familes of compact convex sets.

A family of sets $\mathcal{S}$ has the (p, q)-*property*, for $p \geq q > 1$ if, for any p sets in $\mathcal{S}$, one may find q among them with nonempty intersection.

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Alon-Kleitman (p, q) theorem (1992)

Let $p \geq q \geq d+1$. There exists n such that any finite family of convex subsets of $\mathbb{R}^d$ with the (p, q)-property has a transversal of size at most n.

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In their celebrated proof Alon and Kleitman used the *fractional Helly theorem for convex sets* (Katchalski-Liu 1979).

II. VC classes

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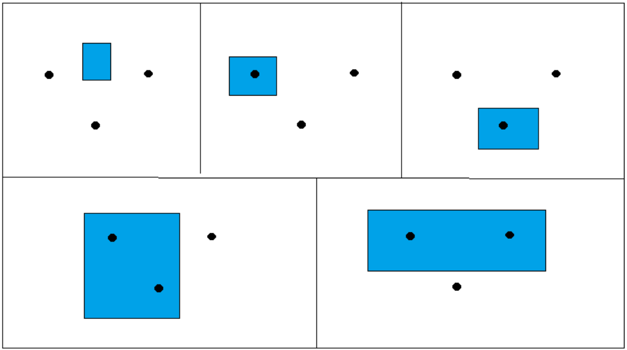
We say that $\mathcal{S}$ *shatters* F if $\mathcal{S} \cap F = \mathcal{P}(F)$.



$\mathcal{S} = \{\text{rectangles}\}$ shatters a set of three points.

Let $\mathcal{S}$ be a family of sets.

The *VC-dimension* of $\mathcal{S}$, denoted $\mathrm{VC}(\mathcal{S})$, is the maximum cardinality of a finite set shattered by $\mathcal{S}$ if it exists. Otherwise $\mathrm{VC}(\mathcal{S}) = \infty$.

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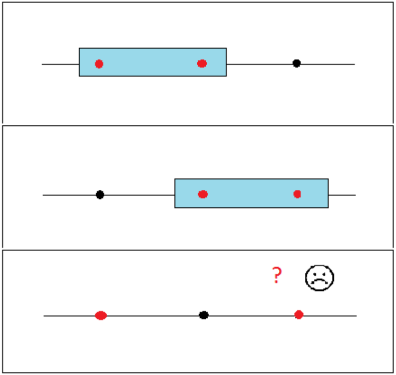

$\mathcal{S} = \{\text{intervals}\}$ fails to shatter any 3 points.

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The *shatter function* $\pi_{\mathcal{S}}(n) : \omega \to \omega$ of $\mathcal{S}$ is given by

$$\pi_{\mathcal{S}}(n) = \max\{|\mathcal{S} \cap F| : |F| = n\}.$$

E.g. $\pi_{\text{rectangles}}(3) = 2^3 = 8.$
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Observe that the VC-dimension of a VC class $\mathcal{S}$ is the maximum n such that $\pi_{\mathcal{S}}(n) = 2^n$.

Sauer's Lemma

If $\mathrm{VC}(\mathcal{S}) \leq k$ then

$$\pi_{\mathcal{S}}(n) \leq \sum_{i=0}^{k} \binom{n}{i} = \mathcal{O}(n^k).$$

Bound is tight: consider $\mathcal{S}$ all subsets of $\{x_1, \ldots, x_n\}$ of cardinality $\leq k$.

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The *VC-density* of a VC class $\mathcal{S}$, denoted $vc(\mathcal{S})$ is defined as

$$vc(\mathcal{S}) = \inf\{r > 0 : \pi_{\mathcal{S}}(n) = \mathcal{O}(n^r)\}.$$

Who proved Sauer's Lemma?

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- First published proof: Vapnik and Chervonenkis 1971.

  Vladimir N Vapnik and A Ya Chervonenkis. *On the uniform convergence of relative frequencies of events to their probabilities.* Theory of Probability and Its Applications, 16(2):264{280, 1971.

Vapnik and Chervonenkis were studying probability. The lemma is the seminal result in VC theory, an area of learning theory (machine learning).

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- Published by Sauer in 1972.

JOURNAL OF COMBINATORIAL THEORY (A) 13, 145–147 (1972)

On the Density of Families of Sets

N. SAUER

Department of Mathematics, The University of Calgary, Calgary 44, Alberta, Canada

Communicated by Bruce Rothschild

Received February 4, 1970

Sauer was solving an Erdös' puzzle.

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In Sauer's paper:

[1] The referee of this paper wrote that these results have also been established by S. Shelah [1, 2].

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- The lemma appears in Shelah's 1971-72 papers.

  S. Shelah. *Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory.* Ann. Math. Logic 3 (1971), no. 3, 271-362.

  S. Shelah. *A combinatorial problem; stability and order for models and theories in infinitary languages.* Pacific J. Math. 41 (1972), 247-261.

Shelah is doing model theory.

The result is difficult to find in the "thicket of mathematical logic".

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DEFINITION 1.5. $P4(\lambda, \mu, \chi)$ holds if whenever $|S| = \lambda$, $|A| = \mu$, and S is a family of subsets of A, there exists $B \subset A$, $|B| = \chi$, such that for every $C \subset B$ there is $X \in S$ such that $X \cap B = C$.

Clearly $P4(\lambda, \mu, \chi)$ implies $P3(\lambda, \mu, \chi)$ and $P3(\lambda, \mu, \alpha)$ for every $\alpha < \chi^+$. The only result known to me is that if $\lambda \geqq \mathrm{Ded}(\mu)$, λ is regular and χ is finite, then $P_4(\lambda, \mu, \chi)$ holds. (see Shelah [15]). Perles and I prove that if μ and χ are finite $P4(\lambda, \mu, \chi)$ holds if and only if $\lambda > \sum_{k=0}^{\chi-1} \binom{\mu}{k}$. Later and independently Sauer [19] proved it.

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The (Vapnik-Chervonenkis-Shelah-Perles)-Sauer Lemma

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M. J. Steele (UPenn):

- “I learned the VC lemma from their 1971 paper. I mentioned this to Erdös in 1973 or 1974 and he told me about Sauer and Shelah. [. . .] Erdös definitely thought at that time that Sauer and Shelah were the first to answer his question [. . .]. Incidentally, I think Erdös spoke more affectionately about Shelah than any other mathematician he ever mentioned to me.”

Pajor's Lemma (1985)

Any finite family of sets $\mathcal{F}$ shatters at least $|\mathcal{F}|$ sets.

Sauer's Lemma follows if you consider its contrapositive:

$$\pi_{\mathcal{S}}(n) = |\mathcal{S} \cap F| > \sum_{i=0}^{k} \binom{n}{i} \Rightarrow \mathrm{VC}(\mathcal{S}) > k.$$

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Induction: Suppose that $|\mathcal{F}| > 1$. Let x be an element in some but not all sets in $\mathcal{F}$. Let

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Clearly, $\mathcal{F}_i$, $i = 0, 1$, does not shatter any set that contains x. Let

$$\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \{S \cup \{x\} : S \in \mathcal{S}_0 \cap \mathcal{S}_1\}.$$

Then $\mathcal{F}$ shatters every set in $\mathcal{S}$ and $|\mathcal{S}| = |\mathcal{F}|$.

The dual system

Given $\mathcal{S}$ a family of subsets of some set X, consider the dual family $\mathcal{S}^*$ of sets of the form

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- The dual shatter function of $\mathcal{S}$ is given by

$$\pi^*_{\mathcal{S}}(n) = \pi_{\mathcal{S}^*}(n).$$

- The VC-codimension and VC-codensity of $\mathcal{S}$ are respectively

$$\mathrm{VC}^*(\mathcal{S}) = \mathrm{VC}(\mathcal{S}^*),$$
$$\mathrm{vc}^*(\mathcal{S}) = \mathrm{vc}(\mathcal{S}^*).$$

Dual family of $\mathcal{S}$: sets of the form $\mathcal{S}_x = \{S \in \mathcal{S} : x \in S\}$ for $x \in X$.

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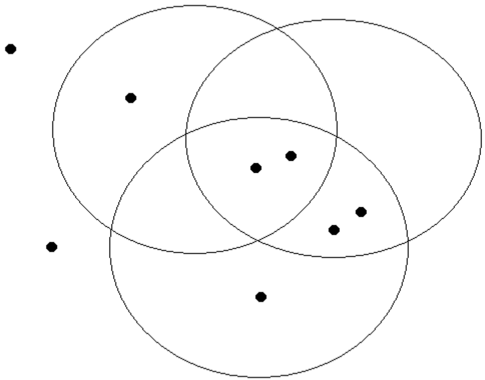

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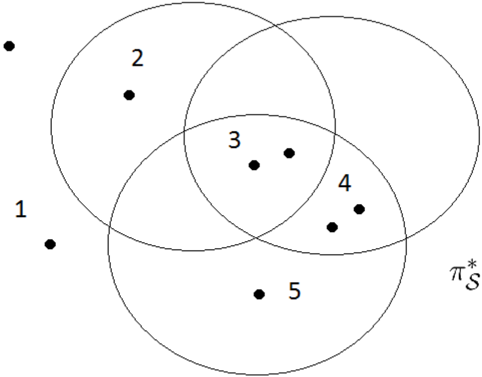

$\pi^*_{\mathcal{S}}(3) = 5.$

For any family of sets we have:

$$\begin{array}{ccc} \text{Finite VC-dimension} & \Longleftrightarrow & \text{Finite VC-density} \\ \Updownarrow & & \Updownarrow \\ \text{Finite VC-codimension} & \Longleftrightarrow & \text{Finite VC-codensity.} \end{array}$$

A (p, q) theorem for VC classes

Alon-Kleitman-Matoušek (p, q) theorem (2004)

Let $\mathcal{S}$ be a VC class. Then, for any integers $p \geq q > vc^*(\mathcal{S})$, there exists some n such that, for any finite $\mathcal{F} \subseteq \mathcal{S}$, if $\mathcal{F}$ has the (p, q)-property, then $\mathcal{F}$ has a transversal of size $\leq n$.

Stronger conclusion: Any subfamily $\mathcal{F} \subseteq \mathcal{S}$ with the (p, q)-property can be partitioned into at most n consistent subfamilies.

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Recall that Alon and Kleitman used a fractional Helly theorem for convex sets to prove their (p, q) theorem.

- Matoušek proved a fractional Helly theorem for VC classes.
- He then observed that the Alon-Kleitman method yielded a (p, q) theorem.

Finding the n (transversal size) given by the (p, q) theorems is a subject of current research.

A family of convex sets in the plane satisfying the $(4, 3)$-property can be pierced by 9 points

Daniel McGinnis

October 27, 2020

2020

From a $(p, 2)$-Theorem to a Tight (p, q)-Theorem

Chaya Keller* Shakhar Smorodinsky†

2017

Abstract

A family $\mathcal{F}$ of sets is said to satisfy the (p, q)-property if among any p sets of $\mathcal{F}$ some q have a non-empty intersection. The celebrated (p, q)-theorem of Alon and Kleitman asserts that any family of compact convex sets in $\mathbb{R}^d$ that satisfies the (p, q)-property for some $q \geq d + 1$, can be pierced by a fixed number (independent on the size of the family)

Improved bounds on the Hadwiger-Debrunner numbers*

Chaya Keller† Shakhar Smorodinsky‡ Gábor Tardos§

2016

Abstract

Let $\mathrm{HD}_d(p, q)$ denote the minimal size of a transversal that can always be guaranteed for a family of compact convex sets in $\mathbb{R}^d$ which satisfy the (p, q)-property $(p \geq q \geq d + 1)$. In a celebrated proof of the Hadwiger-Debrunner conjecture, Alon and Kleitman proved that $\mathrm{HD}_d(p, q)$ exists for all $p \geq q \geq d+1$. Specifically, they prove that $\mathrm{HD}_d(p, d+1)$ is $\tilde{O}(p^{d^2+d})$.

Piercing axis-parallel boxes

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2018

III. Model Theory

- A structure $\mathcal{M}$ on a set M is a collection of *definable* subsets of M^n, for every $n < \omega$. These are closed under boolean operations, projections, and cartesian products; and contain singletons and diagonal sets.
- A function is definable if its graph is a definable set.
- E.g. the structure on a field $(K, +, \cdot)$ is the smallest structure containing the graphs of the sum and product.
- A *definable family of sets* $\{S_a : a \in D\}$ is the collection of fibers of some definable set.

Definition

Given a structure $\mathcal{M}$, a definable family of sets is NIP (not the independence property) if it is a VC class.

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- Shelah (1971): a structure is NIP iff every definable family of unary sets is NIP.
- We known many examples of NIP structures: stable, o-minimal, dp-minimal ...
- Laskowski (1992) publishes a paper on the relationship between VC classes and NIP structures.
  He uses NIP literature to identify new VC classes.

A.K.M. (p, q) theorem (2004)

Let $\mathcal{S}$ be a VC class. Then, for any integers $p \geq q > vc^*(\mathcal{S})$, there exists some n such that, for any subfamily $\mathcal{F} \subseteq \mathcal{S}$ with the (p, q)-property can be partitioned into at most n consistent subfamilies.

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The theorem has found applications in model theory: uniform honest definitions, study of convex sets in valued fields ...

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Corollary (of A.K.M. (p, q) theorem)

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Definable (p, q) conjecture [Simon and Chernikov 2015]

Let $\mathcal{S} = \{S_a : a \in D\}$ be a definable VC class of sets in some structure, and let $p \geq q > vc^*(\mathcal{S})$. If $\mathcal{S}$ has the (p, q)-property then $\mathcal{S}$ can be partitioned into finitely many consistent **definable** subfamilies, i.e. there exists a finite partition of D into definable sets $D_1, \ldots, D_n$, such that for each i the family $\{S_a : a \in D_i\}$ is consistent.

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The study of the notions of forking and dividing in NIP structures led naturally to the following conjecture.

Definable (p, q) conjecture [Simon and Chernikov 2015]

Let $\mathcal{S} = \{S_a : a \in D\}$ be a definable VC class of sets in some structure, and let $p \geq q > vc^*(\mathcal{S})$. If $\mathcal{S}$ has the (p, q)-property then $\mathcal{S}$ can be partitioned into finitely many consistent **definable** subfamilies, i.e. there exists a finite partition of D into definable sets $D_1, \ldots, D_n$, such that for each i the family $\{S_a : a \in D_i\}$ is consistent.

There are some partial proofs, e.g. for distal structures.

There is also a strong form of the conjecture for dp-minimal structures, where the conclusion is that $\mathcal{S}$ partitions into definable subfamilies that extend each to a *definable type*.

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Let $\mathcal{S} = \{S_a : a \in D\}$ be a definable family of sets in a structure and $\mathrm{vc}^*(\mathcal{S}) < 2$. If $\mathcal{S}$ has the $(\omega, 2)$-property then $\mathcal{S}$ can be parititioned into finitely many consistent definable subfamilies.

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Questions:

- Why not ask for a **uniform** definable (p, q) theorem?
- In the A.K.M. (p, q) theorem, can the (p, q)-property be relaxed to the (ω, q)-property?

In 2009 at the Fields Institute, Aschenbrenner and Fischer are trying to prove a result on existence of definable Lipschitz extensions of functions (Definable Kirszbraun's theorem). To do it they need a definable Helly theorem.

Definable Helly theorem [Aschenbrenner-Fischer 2011]

Let $\mathcal{M}$ be a definably complete expansion of a real closed field $(M, +, \cdot, <)$. Let $\mathcal{C}$ be a definable family of closed and bounded convex subsets of M^d. If $\mathcal{C}$ is $(d+1)$-consistent then $\cap\mathcal{C} \neq \emptyset$.

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(2) By (1) $\mathcal{C}$ is consistent.

(3) Let $\mathcal{D}$ be the definable family of all intersections of at most $d+1$ sets in $\mathcal{C}$.

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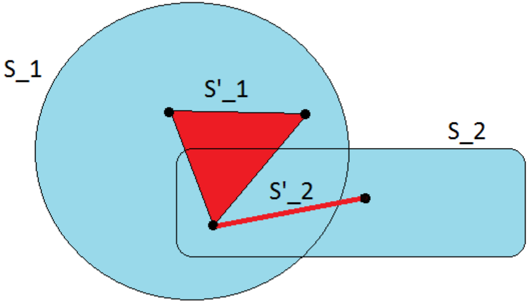

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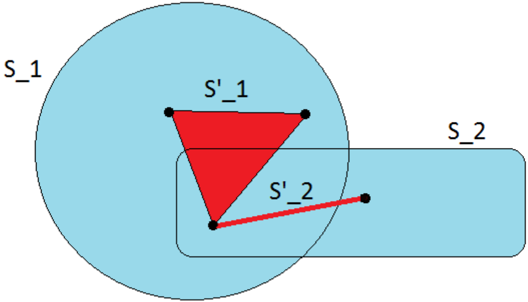

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(8) Applying the finite version again (1), we reach that

$$\emptyset \neq \cap \mathcal{C}' \subseteq \cap \mathcal{C}.$$

Success!

Thank you for listening.