Defining δefinable compactness

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IMPORTANT QUESTION:

How can we capture the notion of topological compactness as a first-order logic property?

Ya'acov Peterzil and Charles Steinhorn. *Definable compactness and definable subgroups of o-minimal groups.* J. London Math. Soc. (2), Volume 59 (3), 1999.

1 Introduction

Compactness plays a crucial role in topological algebra. Since groups and rings definable in o-minimal structures can be equipped definably with (an essentially unique) structure making them, respectively, topological groups and rings (see [Pi], [OPP]), a version of compactness that works in the model-theoretic setting of o-minimality is desirable. [...]

Definable topologies

Definition

A topological space (X, τ) is definable* in a structure M if it has a basis that is (uniformly) definable in M.

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Examples:

- Any finite topology.
- The discrete topology on a definable set.
- The order topology induced by a definable linear order.
- The product of two definable topological spaces is definable (e.g. euclidean topology).
- Definable metric and normed spaces in o-minimal structures (M. Thomas and E. Walsberg).
  For $a, b \in \mathbb{R}$ let $f_{a,b} : [0, 1] \to \mathbb{R}$, $f_{a,b}(x) = ax + b$.
  $d((a, b), (a', b')) = \|f_{a,b} - f_{a',b'}\|_\infty$.

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Other examples [A. Pillay ('87)]:

- The valuation topology in a valued field $(F, +, \cdot, V)$.
- The topology in $(\mathbb{C}, +, \cdot, R)$, where R is a unary predicate for the reals.

Topological compactness in FO logic

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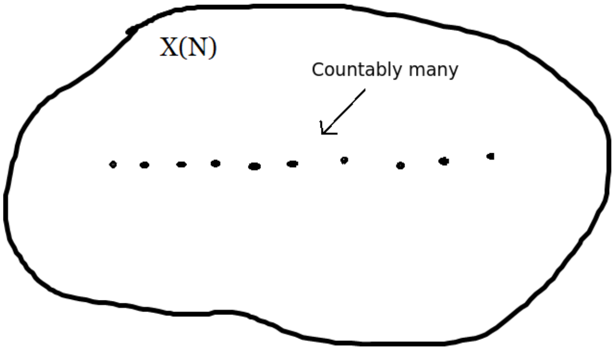
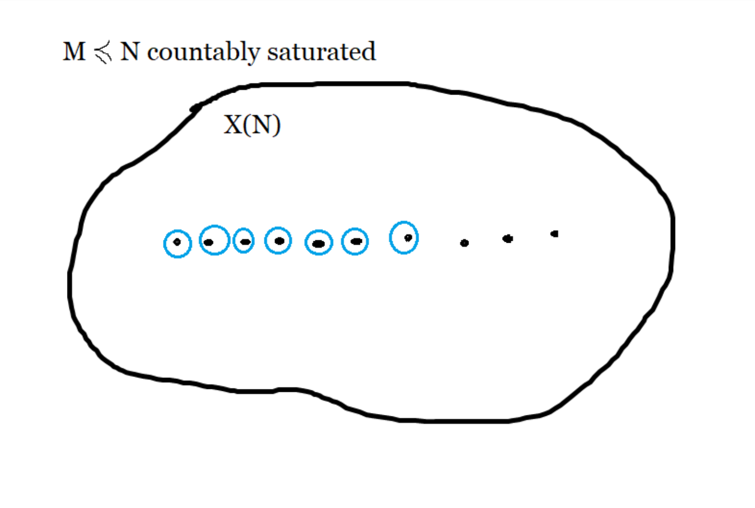
Ideally we want a notion that is maintained under *elementary equivalence*. If $\mathcal{M} \equiv \mathcal{N}$ and $\varphi(v)$ is such that $\varphi(M)$ is definably compact then $\varphi(N)$ should be definably compact too.

Proposition

For any infinite Hausdorff definable topological space in M, compactness is not captured by $\mathrm{Th}(M)$.

M $\preccurlyeq$ N countably saturated

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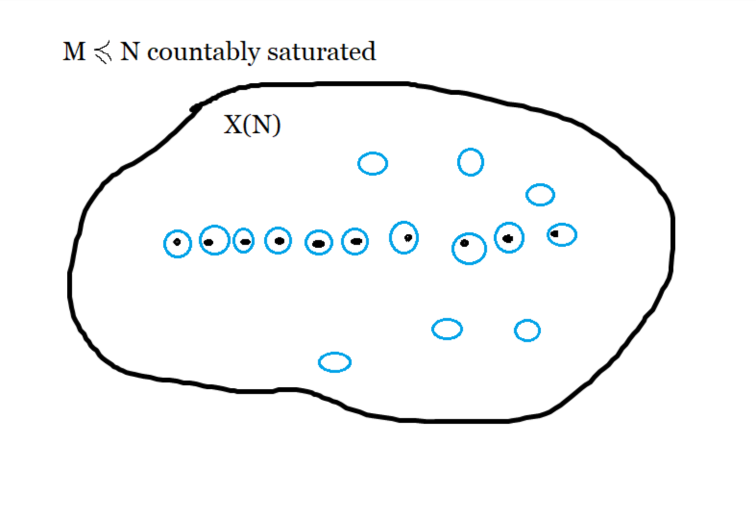
X(N)



Open cover given by ◯ has no finite subcover.

A natural approach

Idea: A topological space (X, τ) is compact if every family of τ-closed sets with the finite intersection property has non-empty intersection.

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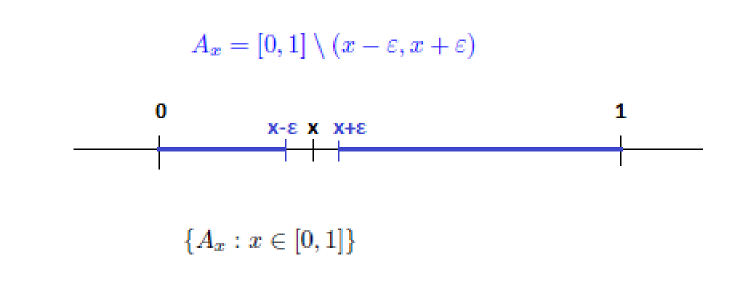
Consider the following example in an elementary extension of the real numbers with infinitesimals (ε).



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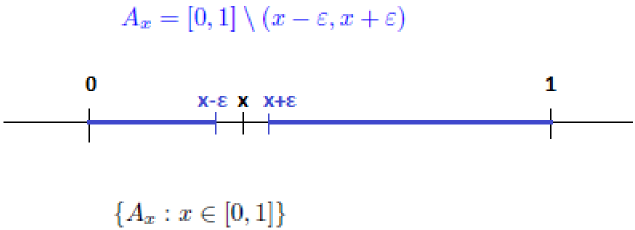

The definable family of (closed and bounded) sets $\{A_x : x \in [0,1]\}$ has the finite intersection property but $\bigcap_{x\in[0,1]} A_x = \emptyset$. So the closed interval $[0,1]$ would not be definably compact.

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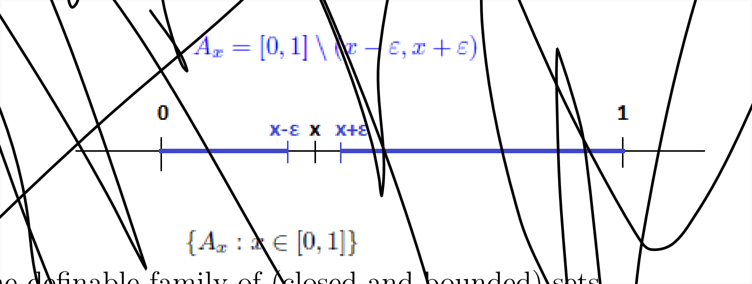

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Some quick conventions:

- Definable means with parameters in general.
- Types = Filters.

Ya'acov Peterzil and Charles Steinhorn. *Definable compactness and definable subgroups of o-minimal groups.* J. London Math. Soc. (2), Volume 59 (3), 1999.

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Curve-compactness

Fix a definable topological space (X, τ) in a structure M expanding a dense linear order.

A definable curve $\gamma : (a, b) \subseteq M \to X$ τ-*converges* to $x \in X$ as $t \to a$ if, for every τ-neighborhood A of x, there exists $c_A \in (a, b)$ such that $\gamma(t) \in A$ for every $t \in (a, c_A)$. τ-Convergence at $t \to b$ is defined analogously.

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Definition (curve-compactness)

(X, τ) is *curve-compact* if every definable curve $\gamma : (a, b) \to X$ τ-converges as $t \to a$ and as $t \to b$ (i.e. γ is τ-completable).

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Proposition (Peterzil-Steinhorn '99)

A definable set in a o-minimal structure (with the canonical o-minimal topology) is definably compact if and only if it is closed and bounded.

Ehud Hrushovski and François Loeser. *Non-Archimedean Tame Topology and Stably Dominated Types*. Annals of Mathematics Studies, 192, 2016.

In Chapter 4 we define the central notion of definable compactness; we give a general definition that may be useful whenever one has definable topologies with enough definable types. The o-minimal formulation regarding limits of curves is replaced by limits of definable types. We relate definable compactness to being closed and bounded. We show the expected properties hold, in particular the image of a definably compact set under a continuous definable map is definably compact.

Type-compactness

Let (X, τ) be a definable topological space in a structure M. A definable type $p \in S_X^{\mathrm{def}}(M)$ τ-converges to a point (τ-limit) $x \in X$ if x belongs in every τ-closed set in p.

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If M is o-minimal, a definable curve $\gamma : (a, b) \to X$ τ-converges as $t \to a$ iff the type generated by sets $\gamma[(a, t)]$ for $a < t < b$ τ-converges.

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Proposition

Let M be either an o-minimal expansion of $(\mathbb{R}, <)$ or $\mathbb{Q}_p$ for some prime p. Then any definable topological space in M is type-compact if and only if it is compact.

mathoverflow

Definitions of definable compactness

Asked 13 years ago Modified 6 months ago Viewed 280 times

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We have an o-minimal structure M with the order topology. $X \subseteq M^n$ with the induced topology. The article "Definable compactness and definable subgroups of o-minimal groups" by Steinhorn and Peterzil shows that $X \subseteq M^n$ is definable compact if and only if X is being bounded and closed.

Definable compactness of X means that any M-definable curve in X is completable. (a curve in X is an M-definable continuous embedding $f : (a, b) \to X$). It is said to be completable if $\lim_{x \downarrow a} f(x)$ and $\lim_{x \uparrow b} f(x)$ exist.)

In a lecture course at Paris VI, I saw another definition of definable compactness: X is definable compact if and only if any M-definable type on X has a limit in X. An M-definable

Why are these two definitions equivalent? I can't find anything on the second definition. Are there any sources easily accessible that explain that equivalence?

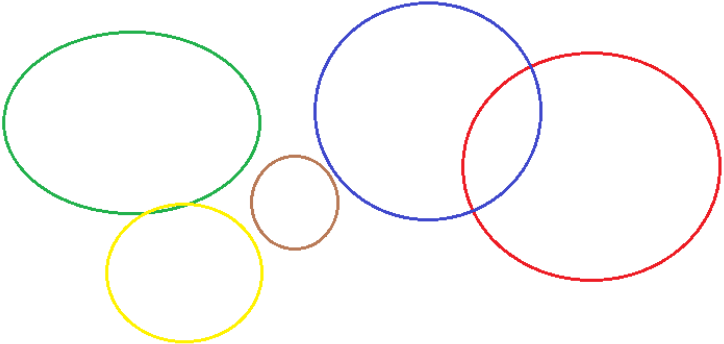
Ya'acov Peterzil and Anand Pillay. *Generic sets in definably compact groups.* Fundamenta Mathematicae, 193, 2007.

COROLLARY 2.2.

(i) *Assume that $\mathcal{F} = \{F_s : s \in S\}$ is a definable family of closed and bounded sets (S is now a definable set), with the finite intersection property. If $\mathcal{F}$ is definable over $\mathcal{M}_0$ then there are finitely many elements $a_1, \ldots, a_k \in \mathcal{M}_0$ such that every F_s contains one of the a_i's.*

Transversal-compactness

A family of sets $\mathcal{C}$ has a finite transversal $F = \{a_1, \ldots, a_n\}$ if every $C \in \mathcal{C}$ satisfies that $C \cap F \neq \emptyset$.



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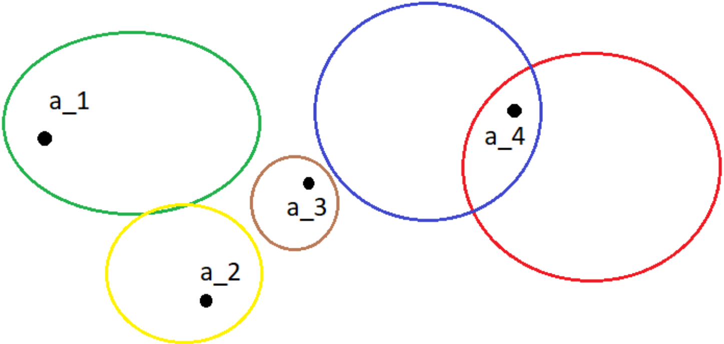

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A topological (X, τ) is *compact* if every consistent (i.e. having the finite intersection property) family of τ-closed sets has a finite transversal.

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This definition seems especially suitable among NIP structures.

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Theorem (I. Kaplan '23)

Let (X, τ) be a definable topological space in an NIP structure. Then (X, τ) is transversal-compact if and only if every definable family $\mathcal{C}$ of τ-closed sets that is k-consistent, for $k > vc^*(\mathcal{C})$, has a finite transversal.

Filter-compactness

A family of sets $\mathcal{C}$ is downward directed if, for every $C_1, C_2 \in \mathcal{C}$, the exists $C_3 \in \mathcal{C}$ with $C_3 \subseteq C_1 \cap C_2$.

Definition (filter-compactness)

A definable topological space (X, τ) is *filter-compact* if every definable downward directed family of nonempty τ-closed sets has non-empty intersection.

This definition has been explored in recent years by A. Fornasiero, W. Johnson and others.

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(i) A definably compact subspace of a Hausdorff space is closed.

(ii) A definable continuous bijection from a definably compact to a Hausdorff space is a homeomorphism.

(iii) In a definably complete ordered field K, any definable continuous function $f : D \to M$, where $D \subseteq K^n$ is definably compact, reaches its maximum and minimum.

O-minimal definable compactness

Theorem (AG '21)

Let (X, τ) be a definable topological space in an o-minimal structure M. TFAE

1. (X, τ) is type-compact.
2. (X, τ) is filter-compact.
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Moreover all the above imply and, if τ is Hausdorff or M has definable choice, are equivalent to:

4. (X, τ) is curve-compact.

Definable compactness in pCF

Theorem (AG and Johnson '22)

Let (X, τ) be a definable topological space in a p-adically closed field $K \models \mathrm{Th}(\mathbb{Q}_p)$. TFAE.

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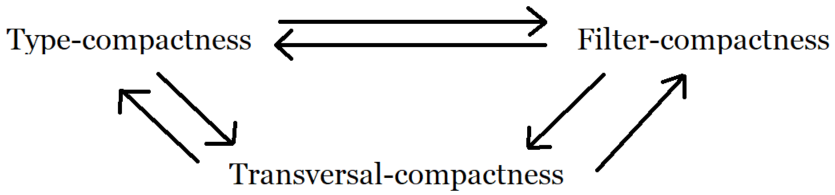
We also came up with suitable notions of definable curve and curve-compactness in this setting, and proved that it is equivalent to (1)-(3) in the above theorem.

Curves: Definable maps $f : D \subseteq K \to X$ with $0 \in cl(D) \setminus D$.

Curve-compactness: Every definable curve in X has a τ-converging definable restriction.


Type-compactness
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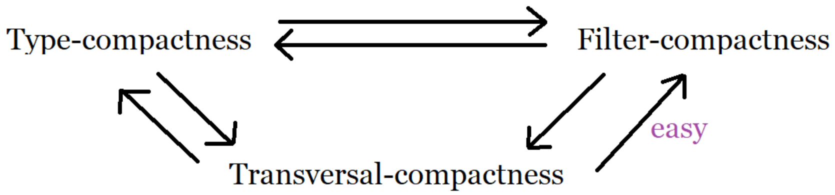
Type-compactness
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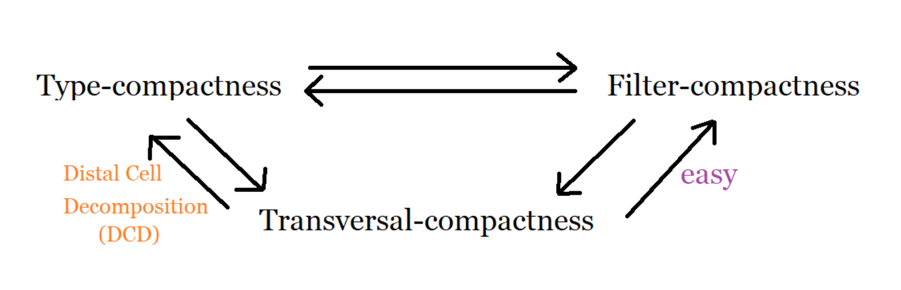
Type-compactness
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Distal Cell
Decomposition
(DCD)
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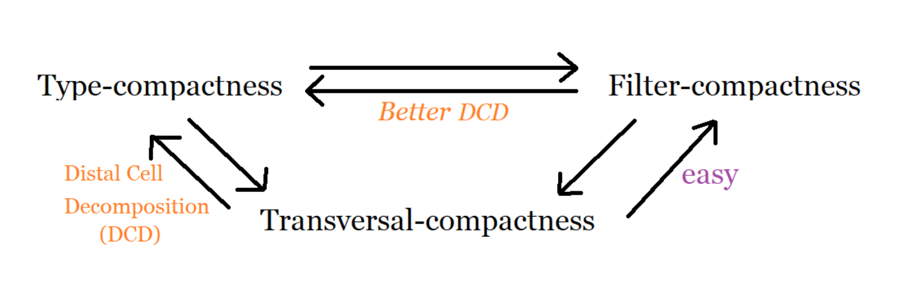
easy

Theorem (A.G. '21, A.G.-Johnson '22):
Better Distal Cell Decomposition

Let M be either **o-minimal** or a p**-adically closed field**. For every type $p(x) \in S^{\mathrm{def}}(M)$ and (partitioned) formula $\varphi(x, y)$, there exists another formula $\psi(x, z)$ such that $p|_{\psi}(x)$ is downward directed and

$$p|_{\psi}(x) \vdash p|_{\varphi}(x).$$

Establishes a connection between definable types and definable downward directed families of sets.


Type-compactness
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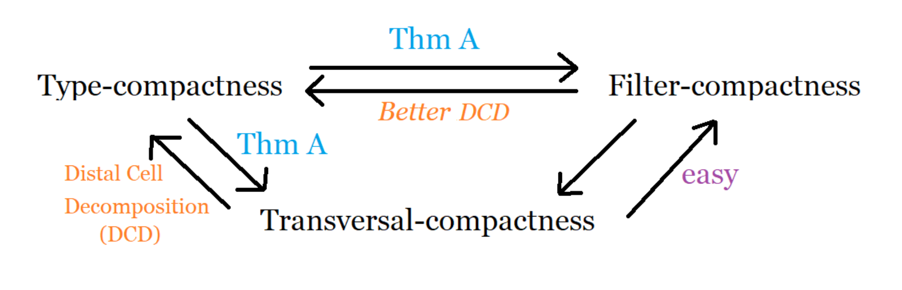
easy

Theorem A (Simon and Starchenko '14)

Let M be a dp-minimal structure with $(*)$. Let $\mathcal{S}$ be a consistent definable family of sets. Then $\mathcal{S}$ can be partitioned into finitely many subfamilies $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that, for each $i \leq k$, $\mathcal{S}_i$ extends to a definable type (complete over M).

Property $(*)$ is satisfied by:

- linearly ordered dp-minimal theories;
- dp-minimal theories with definable Skolem function;
- unpackable VC-minimal theories.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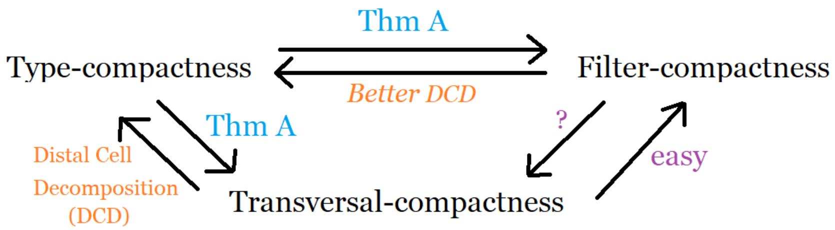
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By proving a weakening of Theorem *Better DCD* (among distal dp-minimal structures) we managed to show:

Theorem (85% confidence)

Let M be a distal, dp-minimal structure with $(*)$. For any definable topological space (X, τ) the following are equivalent:

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Conjectures

1. Theorem A holds all dp-minimal theories. In particular, condition $(*)$ can be dropped in the theorem above.
2. Theorem *Better DCD* holds in all distal dp-minimal structures.

Finitely satisfiable generics (fsg)

Ehud Hrushovski, Ya'acov Peterzil and Anand Pillay. *Groups, measures, and the NIP*. Journal AMS , 21(2), 2006.

4 Groups with finitely satisfiable generics

Here we introduce a certain desirable property of definable groups which we call fsg (standing for "finitely satisfiable generics") In Section 7 of the paper we prove that definably compact groups definable in o-minimal expansions of real closed fields have fsg.

Definition 4.1. G has fsg (*finitely satisfiable generics*) if there is some global type $p(x)$ and some small model M_0 such that $p(x) \models$ "$x \in G$", and every left translate $gp = \{\phi(x) : \phi(g^{-1}x) \in p\}$ of p with $g \in G$, is finitely satisfiable in M_0.

“Definably compact groups in a variety of settings [...] have either been proved to have or are expected to have fsg"
- Hrushovski-Pillay ’11.

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Theorem (Hrushovski-Peterzil-Pillay '06)

A definable group in an o-minimal expansion of an ordered field has fsg if and only if it is definably compact.

Theorem (Johnson '22)

A definable group in a p-adically closed field has fsg if and only if it is definably compact.

Definition

A definable group G is *definably amenable* if it admits a (left) G-invariant Keisler measure.

These include stable groups and definably compact groups in o-minimal structures or in the p-adics.

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Conjecture

Let M be an NIP structure. Any definably amenable group in M that is transversal-compact has fsg.

Thank you.