Types and definable compactness in o-minimality and beyond

Pablo Andújar Guerrero (with Will Johnson)

Fields Institute

Model Theory Seminar

Conventions

- Throughtout $\mathcal{M} = (M, \dots)$ denotes a first order structure.
- Definable means in $\mathcal{M}$ possibly with parameters.
- A (uniformly) definable family $\mathcal{S}$ of subsets of M^n is a family given by some formula $\varphi(x, y)$, $|x| = n$, such that

$$\mathcal{S} = \{\varphi(M^n, a) : a \in M^{|y|}\}.$$

- A family of sets $\mathcal{S}$ is (finitely) consistent if every finite intersection of sets in $\mathcal{S}$ is nonempty.
- Types are complete and over M.
- We adopt the convention of considering types to be consistent families of definable sets. In particular an n-type is an ultrafilter in the boolean algebra of definable subsets of M^n.
- A type $p(x)$ is definable if, for every formula $\varphi(x, y)$, the set $\{a : \varphi(M^{|x|}, a) \in p(x)\}$ is definable.

The main result

A family of sets $\mathcal{S}$ is *downward directed* if, for every $S_0, S_1 \in \mathcal{S}$, there exists $S_2 \in \mathcal{S}$ with $S_2 \subseteq S_0 \cap S_1$, and moreover $\mathcal{S}$ does not contain the empty set.

Given two families of sets $\mathcal{S}$ and $\mathcal{F}$, we say that $\mathcal{F}$ is *finer* than $\mathcal{S}$ if, for every $S \in \mathcal{S}$, there is some $F \in \mathcal{F}$ such that $F \subseteq S$.

The main result

A family of sets $\mathcal{S}$ is *downward directed* if, for every $S_0, S_1 \in \mathcal{S}$, there exists $S_2 \in \mathcal{S}$ with $S_2 \subseteq S_0 \cap S_1$, and moreover $\mathcal{S}$ does not contain the empty set.

Given two families of sets $\mathcal{S}$ and $\mathcal{F}$, we say that $\mathcal{F}$ is *finer* than $\mathcal{S}$ if, for every $S \in \mathcal{S}$, there is some $F \in \mathcal{F}$ such that $F \subseteq S$.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family of sets extends to a definable type.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

The main result

A family of sets $\mathcal{S}$ is *downward directed* if, for every $S_0, S_1 \in \mathcal{S}$, there exists $S_2 \in \mathcal{S}$ with $S_2 \subseteq S_0 \cap S_1$, and moreover $\mathcal{S}$ does not contain the empty set.

Given two families of sets $\mathcal{S}$ and $\mathcal{F}$, we say that $\mathcal{F}$ is *finer* than $\mathcal{S}$ if, for every $S \in \mathcal{S}$, there is some $F \in \mathcal{F}$ such that $F \subseteq S$.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family of sets extends to a definable type.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

It follows that a definable family of sets in an o-minimal structure extends to a definable type iff it admits a finer downward directed family.

The main result

A family of sets $\mathcal{S}$ is *downward directed* if, for every $S_0, S_1 \in \mathcal{S}$, there exists $S_2 \in \mathcal{S}$ with $S_2 \subseteq S_0 \cap S_1$, and moreover $\mathcal{S}$ does not contain the empty set.

Given two families of sets $\mathcal{S}$ and $\mathcal{F}$, we say that $\mathcal{F}$ is *finer* than $\mathcal{S}$ if, for every $S \in \mathcal{S}$, there is some $F \in \mathcal{F}$ such that $F \subseteq S$.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family of sets extends to a definable type.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

We can use Theorem A to characterize a notion of definable compactness for o-minimal definable topologies.

Definable topologies

Definition (Definable topological space)

A topological space (X, τ) is definable if it has a basis that is definable.

Definable topologies

Definition (Definable topological space)

A topological space (X, τ) is definable if it has a basis that is definable.

Examples in o-minimal structures:

- The o-minimal "euclidean" topology.
- Definable manifold spaces (e.g. definable groups [Pillay 1988]).
- Some definable spaces of C^r funtions with r-norm [Thomas 2012].
- Definable metric spaces [Walsberg 2015].
- The Split interval, Alexandrov double circle, Moore plane ...

Other examples [A. Pillay (1987)]:

- The valuation topology in a valued field $(F, +, \cdot, V)$.
- The topology in $(\mathbb{C}, +, \cdot, P)$, where P is a unary predicate for the reals.

Defining Definable Compactness

Definition (Definable compactness)

A definable topological space (X, τ) is *definably compact* if every downward directed definable family of closed sets has nonempty intersection.

This definition has received attention in recent years by Fornasiero and Johnson.

Defining Definable Compactness

Definition (Definable compactness)

A definable topological space (X, τ) is *definably compact* if every downward directed definable family of closed sets has nonempty intersection.

This definition has received attention in recent years by Fornasiero and Johnson.

Some facts:

- The image of a definably compact space by a definable continuous function is definably compact.
- In a definably complete field $\mathcal{M}$, any definable continuous function $f : K \to M$, where K is definably compact, reaches its maximum and minimum.
- Definable compactness allows us to generalize results on finite families of definable sets to infinite definable families.

Type-compactness

Let p be a type in some definable topological space (X, τ), with $X \in p$. Say that $x \in X$ is a *limit* of p (in (X, τ)) if $x \in C$ for every closed set $C \in p$.

Type-compactness

Let p be a type in some definable topological space (X, τ), with $X \in p$. Say that $x \in X$ is a *limit* of p (in (X, τ)) if $x \in C$ for every closed set $C \in p$.

Definition (type-compactness)

A definable topological space (X, τ) is *type-compact* if every definable type p with $X \in p$ has a limit in X.

This notion was considered by Hrushovski and Loeser (2016) in their book "Non-Archimedean Tame Topology and Stably Dominated Types".

Type-compactness

Let p be a type in some definable topological space (X, τ), with $X \in p$. Say that $x \in X$ is a *limit* of p (in (X, τ)) if $x \in C$ for every closed set $C \in p$.

Definition (type-compactness)

A definable topological space (X, τ) is *type-compact* if every definable type p with $X \in p$ has a limit in X.

This notion was considered by Hrushovski and Loeser (2016) in their book "Non-Archimedean Tame Topology and Stably Dominated Types".

Lemma 1

Let (X, τ) be a definable topological space. TFAE

(1) (X, τ) is type-compact.

(2) Every definable family $\mathcal{C}$ of closed subsets of X that extends to a definable type satisfies that $\cap\mathcal{C} \neq \emptyset$.

Defining definable compactness

Definition (Definable compactness)

A definable topological space (X, τ) is *definably compact* if every definable downward directed family of nonempty closed sets has nonempty intersection.

Definition (type-compactness)

A definable topological space (X, τ) is *type-compact* if every definable type p with $X \in p$ has a limit in X.

Defining definable compactness

Definition (Definable compactness)

A definable topological space (X, τ) is *definably compact* if every definable downward directed family of nonempty closed sets has nonempty intersection.

Definition (type-compactness)

A definable topological space (X, τ) is *type-compact* if every definable type p with $X \in p$ has a limit in X.

Definition (Curve-compactness [Peterzil and Steinhorn 1996])

A definable topological space (X, τ) is *curve-compact* if every *definable curve* $\gamma : (a, b) \to X$ is completable, which means that $\lim_{x \to a^+} \gamma(x)$ and $\lim_{t \to b^-} \gamma(x)$ both exist.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family extends to a definable type.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family extends to a definable type.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

Corollary 2 [AG 2021+]

Let $\mathcal{M}$ be o-minimal and let (X, τ) be a definable topological space. TFAE.

(1) (X, τ) is definably compact.

(2) (X, τ) is type-compact, i.e. every definable family of closed subsets of X that extends to a definable type has nonempty intersection.

Corollary 2 [AG 2021+]

Let $\mathcal{M}$ be o-minimal and let (X, τ) be a definable topological space. TFAE.

(1) (X, τ) is definably compact.

(2) (X, τ) is type-compact, i.e. every definable family of closed subsets of X that extends to a definable type has nonempty intersection.

Proof

(2)$\Rightarrow$(1): Let $\mathcal{C}$ be a definable downward directed family of closed sets. Then $\mathcal{C}$ extends to a definable type, and so $\cap\mathcal{C} \neq \emptyset$.

(1)$\Rightarrow$(2): Let $\mathcal{C}$ be a definable family of closed sets that extends to a definable type. Let $\mathcal{F}$ be a finer definable downward directed family. Consider $\mathcal{F}' = \{cl(F) : F \in \mathcal{F}\}$. By definable compactness there exists $x \in \cap\mathcal{F}'$. Clearly $x \in \cap\mathcal{C}$.

Let (D) denote the union of the following classes of *dp*-minimal theories:

- linearly ordered dp-minimal theories;
- unpackable VC-minimal theories;
- dp-minimal theories with definable Skolem function.

Theorem 3 [Simon and Starchenko 2014]

Suppose that $Th(\mathcal{M})$ is in (D). Let $\mathcal{S}$ be a consistent definable family of sets. Then $\mathcal{S}$ can be partitioned into finitely many subfamilies, each of which extends to a definable type.

Let (D) denote the union of the following classes of *dp*-minimal theories:

- linearly ordered dp-minimal theories;
- unpackable VC-minimal theories;
- dp-minimal theories with definable Skolem function.

Theorem 3 [Simon and Starchenko 2014]

Suppose that $Th(\mathcal{M})$ is in (D). Let $\mathcal{S}$ be a consistent definable family of sets. Then $\mathcal{S}$ can be partitioned into finitely many subfamilies, each of which extends to a definable type.

Given a family of sets $\mathcal{S}$, a set T is a *transversal* for $\mathcal{S}$ if $T \cap S \neq \emptyset$ for every $S \in \mathcal{S}$.

Corollary 4

Suppose that $Th(\mathcal{M})$ is in (D). Let $\mathcal{C}$ be a consistent definable family of closed sets in some type-compact definable topology. Then $\mathcal{C}$ has a finite transversal.

Theorem 5 (Characterization of definable compactness [AG 2021+])

Suppose that $\mathcal{M}$ is o-minimal. Let (X, τ) be a definable topological space. TFAE.

(1) (X, τ) is definably compact.

(2) (X, τ) is type-compact.

(3) Any consistent definable family of closed sets has a finite transversal.

Moreover all the above imply and, if τ is Hausdorff or $\mathcal{M}$ has definable choice, are equivalent to:

(4) (X, τ) is curve-compact.

Back to the main theorem

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family extends to a definable type.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family extends to a definable type.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family extends to a definable type.

Corollary (of Simon-Starchenko) 6

Suppose that $Th(\mathcal{M})$ belongs in (D). Then every downward directed definable family of sets extends to a definable type.

Proof: By Simon-Starchenko (Theorem 3) if $\mathcal{S}$ is downward directed there exist finitely many definable types $p_1, \ldots, p_n$ such that every $S \in \mathcal{S}$ belongs in some p_i. We claim that $\mathcal{S} \subseteq p_i$ for some i. Otherwise there exists, for every i, some $S(i) \in \mathcal{S}$ with $S(i) \notin p_i$. By downward directedness there exists $S \in \mathcal{S}$ with $S \subseteq S(1) \cap \cdots \cap S(n)$. But then $S \notin p_i$ for all i. Contradiction.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(1) Every downward directed definable family extends to a definable type.

In the o-minimal case the proof of Theorem A (1) relies on two facts:

- Small boundaries: for any definable set X it holds that $\dim(cl(X) \setminus X) < \dim X$.
- For any definable family $\mathcal{S}$ there exists n such that every $S \in \mathcal{S}$ has at most n definably connected components.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

Say that a family of sets $\mathcal{S}$ is *redivisible* if the intersection of any two sets in $\mathcal{S}$ is a finite union of sets from $\mathcal{S}$ (e.g. all intervals in a linear order).

Key observation: the restriction of a type to a redivisible family of definable sets is always downward directed.

Proposition 7 [Johnson]

If $\mathcal{M}$ is o-minimal, then for any definable family of sets $\mathcal{S}$ there exists a redivisible definable family of cells $\mathcal{C}$ such that every set in $\mathcal{S}$ is a finite union of sets from $\mathcal{C}$.

In particular, if $\mathcal{S}$ extends to a definable type p, then the restriction of p to $\mathcal{C}$ is going to be a definable downward directed family finer than $\mathcal{S}$.

Theorem A [AG 2021+]

Let $\mathcal{M}$ be o-minimal. The following hold.

(2) For any definable type p and definable family of sets $\mathcal{S} \subseteq p$ there exists a downward directed definable family $\mathcal{F} \subseteq p$ finer than $\mathcal{S}$.

For a (partial) function $f : M^n \to M$ let

$$(-\infty, f) = \{\langle x, t \rangle : x \in dom(f),\ t < f(x)\}.$$

The key fact in proving Theorem A (2) is that o-minimality allows reducing the question to the case where $\mathcal{S}$ is a family of the form $\{(-\infty, f_a) : a \in M^m\}$, and these sets are simple in the following sense.

For any $S = (-\infty, f_a)$ and $S' = (-\infty, f_b)$, and any $x \in dom(f_a) \cap dom(f_b)$, the sets $\{(-\infty, f_a(x)), (-\infty, f_b(x))\}$ are nested. In other words for any $x \in \pi(S \cap S')$, the fibers $\{S_x, S'_x\}$ are nested.

Beyond o-minimality

Conjecture 8

Suppose that there exist a collection of formulas $\{\varphi_i(x, y_i) : i \in I\}$, with $|x| = 1$, such that, for every $i \in I$, the family $\{\varphi_i(M, a) : a \in M^{|y_i|}\}$ is nested. Suppose that, in any model $\mathcal{N} = (N, \ldots)$ of $Th(\mathcal{M})$, any unary definable set is a boolean combination of sets of the form $\varphi_i(N, a)$, $a \in N^{|y_i|}$.

Then Theorem A (2) holds in $\mathcal{M}$, i.e. every definable family of sets that extends to a definable type can be refined to a downward directed family.

This conjecture would apply to weakly o-minimal theories (e.g. take all formulas defining left unbounded intervals).

p-adically closed fields

A p-adically closed field is a model of $Th(\mathbb{Q}_p)$.
Johnson and Yao recently proved the following.

Proposition 9 [Johnson and Yao 2021]

Let $\mathcal{M}$ be a p-adically closed field. A subspace Y of a definable manifold space is definably compact iff any 1-dimensional definable type p with $Y \in p$ has a limit in Y.

p-adically closed fields

A p-adically closed field is a model of $Th(\mathbb{Q}_p)$.
Johnson and Yao recently proved the following.

Proposition 9 [Johnson and Yao 2021]

Let $\mathcal{M}$ be a p-adically closed field. A subspace Y of a definable manifold space is definably compact iff any 1-dimensional definable type p with $Y \in p$ has a limit in Y.

Conjecture 10 [Johnson]

Theorem A (2) holds in p-adically closed fields. In particular definable compactness and type-compactness are equivalent for all definable topological spaces.

Distal cell decomposition

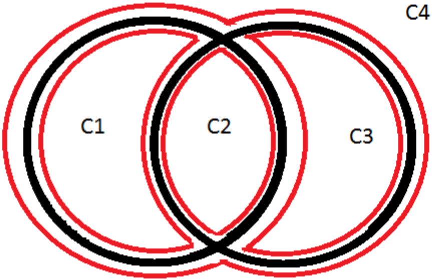
Let $\mathcal{F}$ be a finite family of definable subsets of M^n. An *abstract cell decomposition* for $\mathcal{F}$ is a finite family $\mathcal{C}$ of definable subsets of M^n satisfying that

1. $M^n = \bigcup \mathcal{C}$.
2. For any $F \in \mathcal{F}$ and $C \in \mathcal{C}$, either $C \subseteq F$ or $C \cap F = \emptyset$.

Distal cell decomposition

Let $\mathcal{F}$ be a finite family of definable subsets of M^n. An *abstract cell decomposition* for $\mathcal{F}$ is a finite family $\mathcal{C}$ of definable subsets of M^n satisfying that

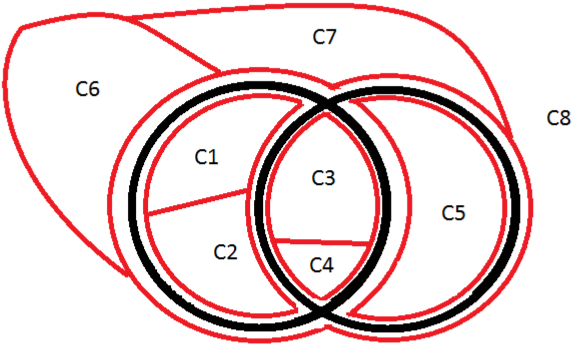
1. $M^n = \bigcup \mathcal{C}$.
2. For any $F \in \mathcal{F}$ and $C \in \mathcal{C}$, either $C \subseteq F$ or $C \cap F = \emptyset$.



Distal cell decomposition

Let $\mathcal{F}$ be a finite family of definable subsets of M^n. An *abstract cell decomposition* for $\mathcal{F}$ is a finite family $\mathcal{C}$ of definable subsets of M^n satisfying that

1. $M^n = \bigcup \mathcal{C}$.
2. For any $F \in \mathcal{F}$ and $C \in \mathcal{C}$, either $C \subseteq F$ or $C \cap F = \emptyset$.



Distal cell decomposition

Let $\mathcal{F}$ be a finite family of definable subsets of M^n. An *abstract cell decomposition* for $\mathcal{F}$ is a finite family $\mathcal{C}$ of definable subsets of M^n satisfying that

1. $M^n = \bigcup \mathcal{C}$.
2. For any $F \in \mathcal{F}$ and $C \in \mathcal{C}$, either $C \subseteq F$ or $C \cap F = \emptyset$.

Distal cell decomposition

Let $\mathcal{M}$ be distal (e.g. weakly o-minimal o pCF). For any definable family of sets $\mathcal{S}$ in $\mathcal{M}$ there exists a definable family $\mathcal{D}$ satisfying that, for any finite $\mathcal{F} \subseteq \mathcal{S}$, there exists an abstract cell decomposition for $\mathcal{F}$ given by some $\mathcal{C} \subseteq \mathcal{D}$.
Let us call $\mathcal{D}$ a **distal cell decomposition** for $\mathcal{S}$.

Distal cell decomposition

Distal cell decomposition

Let $\mathcal{M}$ be distal (e.g. weakly o-minimal o pCF). For any definable family of sets $\mathcal{S}$ in $\mathcal{M}$ there exists a definable family $\mathcal{D}$ satisfying that, for any finite $\mathcal{F} \subseteq \mathcal{S}$, there exists an abstract cell decomposition for $\mathcal{F}$ given by some $\mathcal{C} \subseteq \mathcal{D}$.
Let us call $\mathcal{D}$ a **distal cell decomposition** for $\mathcal{S}$.

Observe: any finite nonempty intersection of sets in $\mathcal{S}$ is going to be a finite union of sets in $\mathcal{D}$.

Distal cell decomposition

Distal cell decomposition

Let $\mathcal{M}$ be distal (e.g. weakly o-minimal o pCF). For any definable family of sets $\mathcal{S}$ in $\mathcal{M}$ there exists a definable family $\mathcal{D}$ satisfying that, for any finite $\mathcal{F} \subseteq \mathcal{S}$, there exists an abstract cell decomposition for $\mathcal{F}$ given by some $\mathcal{C} \subseteq \mathcal{D}$.
Let us call $\mathcal{D}$ a **distal cell decomposition** for $\mathcal{S}$.

Observe: any finite nonempty intersection of sets in $\mathcal{S}$ is going to be a finite union of sets in $\mathcal{D}$.

However $\mathcal{D}$ is not necessarily redivisible.

We can build a sequence of families $\mathcal{D}_1, \mathcal{D}_2, \ldots$, where $\mathcal{D}_j$ is a distal cell decomposition of $\mathcal{D}_i$ for $i < j$, but there is no assurance that this sequence will become constant.

Two observations

Condition: $\mathcal{M}$ is distal **and** there exists finitely many definable families $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of subsets of M such every definable subset of M is a finite boolean combination of sets from these families.

E.g. $\mathcal{M}$ o-minimal, with $\mathcal{S}_1 = \{(-\infty, t] : t \in M\}$ and $\mathcal{S}_2 = \{(-\infty, t) : t \in M\}$.

Two observations

Condition: $\mathcal{M}$ is distal **and** there exists finitely many definable families $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets of M such every definable subset of M is a finite boolean combination of sets from these families.

E.g. $\mathcal{M}$ o-minimal, with $\mathcal{S}_1 = \{(-\infty, t] : t \in M\}$ and $\mathcal{S}_2 = \{(-\infty, t) : t \in M\}$.

Let $\mathcal{D}$ a distal cell decomposition for $\cup_i \mathcal{S}_i$. Then $\mathcal{D}$ contains an abstract cell decomposition for **any** finite family of definable subsets of M. In particular $\mathcal{D}$ is redivisible. So Theorem A (2) holds.

Two observations

To prove that definable compactness implies type-compactness, it suffices to show the existence of a redivisible definable family $\mathcal{D}$ such that every basic closed set is a finite union of sets in $\mathcal{D}$.

Two observations

To prove that definable compactness implies type-compactness, it suffices to show the existence of a redivisible definable family $\mathcal{D}$ such that every basic closed set is a finite union of sets in $\mathcal{D}$.

Suppose that $\mathcal{M}$ expands a linear order $(M, <)$, and that the collection of all definable $<$-convex sets is definable. Then, for any definable $X \subseteq M$, if the order topology on X is definably compact then it is type-compact.

Questions

1. Does Theorem A hold in all distal structures?
2. What characterization of definable compactness is available in other generalizations of o-minimality?
   E.g.
   - Structures with o-minimal open core.
   - d-minimal, noiseless, type A structures.

Thank you for listening.

Pablo And'ujar Guerrero. *Types, transversals and definable compactness in o-minimal structures*. arXiv preprints, 2111.03802, 2021.

Antongiulio Fornasiero. *Definable compactness for topological structures*. In preparation.

Ehud Hrushovski and François Loeser. *Non-Archimedean Tame Topology and Stably Dominated Types*. Annals of Mathematics Studies, 192, 2016.

Will Johnson and Ningyuan Yao. *On non-compact p-adic definable groups*. arXiv preprints, 2103.12427, 2021.

Ya'acov Peterzil and Charles Steinhorn. *Definable compactness and definable subgroups of o-minimal groups*. J. London Math. Soc. (2), Volume 59 (3): 769–786, 1999.

Pierre Simon and Sergei Starchenko. *On forking and definability of types in some DP-minimal theories*. J. Symb. Log., Volume 79 (4) : 1020–1024, 2014.