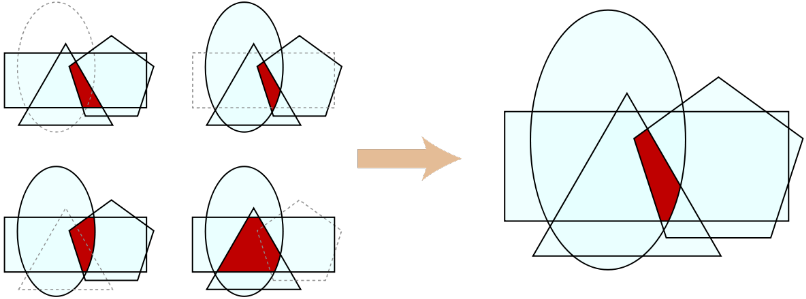
A brief story of (model-theoretic) (p, q)-theorems

Pablo Andújar Guerrero

University of Leeds

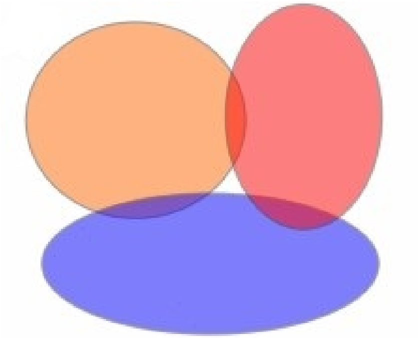
Helly's theorem (1913)

If a finite family $\mathcal{C}$ of convex subsets of $\mathbb{R}^d$ is $(d+1)$-consistent then $\cap\mathcal{C} \neq \emptyset$.



Helly's theorem (1913)

If a finite family $\mathcal{C}$ of convex subsets of $\mathbb{R}^d$ is $(d+1)$-consistent then $\cap\mathcal{C} \neq \emptyset$.



Hadwinger and Debrunner (1957): what if we weaken the assumption and the conclusion of Helly's theorem?

Hadwinger and Debrunner (1957): what if we weaken the assumption and the conclusion of Helly's theorem?

A family of sets $\mathcal{C}$ has the *(p, q)-property*, for $p \geq q \geq 1$ if,

For any p sets in $\mathcal{C}$,	$\forall \mathcal{P} \subseteq \mathcal{C}$, $\lvert\mathcal{P}\rvert = p$,
there exist q many among them,	$\exists \mathcal{Q} \subseteq \mathcal{P}$, $\lvert\mathcal{Q}\rvert = q$,
with nonempty intersection.	$\cap \mathcal{Q} \neq \emptyset$
(also $\emptyset \notin \mathcal{C}$).	

The (q, q)-property equals q-consistency.
As you increase p, the (p, q)-property weakens.

Hadwinger and Debrunner (1957): what if we weaken the assumption and the conclusion of Helly's theorem?

A family of sets $\mathcal{C}$ has the *(p,q)-property*, for $p \geq q \geq 1$ if,

For any p sets in $\mathcal{C}$,	$\forall \mathcal{P} \subseteq \mathcal{C}$, $\lvert\mathcal{P}\rvert = p$,
there exist q many among them,	$\exists \mathcal{Q} \subseteq \mathcal{P}$, $\lvert\mathcal{Q}\rvert = q$,
with nonempty intersection.	$\cap\mathcal{Q} \neq \emptyset$
(also $\emptyset \notin \mathcal{C}$).	

The (q,q)-property equals q-consistency.
As you increase p, the (p,q)-property weakens.

A set T is a *transversal* of $\mathcal{C}$ if

$$T \cap C \neq \emptyset \text{ for every } C \in \mathcal{C}.$$

$\cap\mathcal{C} \neq \emptyset$ means $\mathcal{C}$ has a transversal of size 1.
$\mathcal{C}$ has a transversal of size $n > 1$ is weaker.

Conjecture (Hadwinger and Debrunner 1957)

For any $p \geq d \geq 1$ there exists an $n = n(p, d)$ such that any finite family of convex subsets of $\mathbb{R}^d$ with the $(p, d+1)$-property has a transversal of size $\leq n$.

Conjecture (Hadwinger and Debrunner 1957)

For any $p \geq d \geq 1$ there exists an $n = n(p, d)$ such that any finite family of convex subsets of $\mathbb{R}^d$ with the $(p, d+1)$-property has a transversal of size $\leq n$.

Equivalently: Any family (possibly infinite) of convex subsets of $\mathbb{R}^d$ with the $(p, d+1)$-property can be partitioned into $\leq n$ consistent subfamilies.

(consistent = having the finite intersection property)

~~Conjecture (Hadwinger and Debrunner 1957)~~
Alon-Kleitman (p,q)-theorem (1992)

For any $p \geq d \geq 1$ there exists an $n = n(p,d)$ such that any finite family of convex subsets of $\mathbb{R}^d$ with the $(p, d+1)$-property has a transversal of size $\leq n$.

Equivalently: Any family (possibly infinite) of convex subsets of $\mathbb{R}^d$ with the $(p, d+1)$-property can be partitioned into $\leq n$ consistent subfamilies.

In their celebrated proof Alon and Kleitman used the *fractional Helly theorem for convex sets* (Katchalski-Liu 1979).

Now let us start talking about VC classes

Now let us start talking about VC classes

Let $\mathcal{F}$ be a finite family of subsets of a set X.

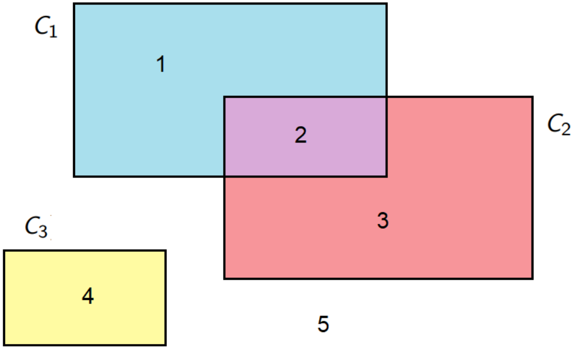
Let $|S_{\mathcal{F}}|$ denote the number of Boolean atoms of $\mathcal{F}$.
(i.e. the number of non-empty areas in the Venn diagram generated by $\mathcal{F}$).

Now let us start talking about VC classes

Let $\mathcal{F}$ be a finite family of subsets of a set X.

Let $|S_{\mathcal{F}}|$ denote the number of Boolean atoms of $\mathcal{F}$. (i.e. the number of non-empty areas in the Venn diagram generated by $\mathcal{F}$).

$\mathcal{F} = \{C_1, C_2, C_3\}$ $\quad |S_{\mathcal{F}}| = 5$

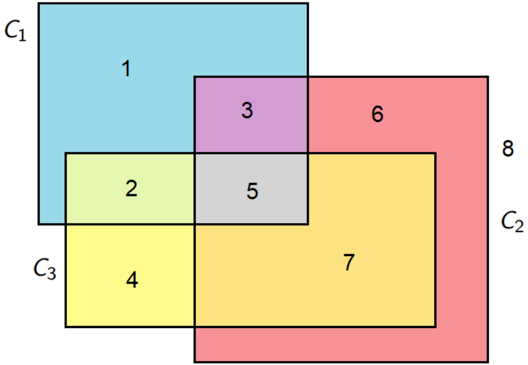


Now let us start talking about VC classes

Let $\mathcal{F}$ be a finite family of subsets of a set X.

Let $|S_{\mathcal{F}}|$ denote the number of Boolean atoms of $\mathcal{F}$.
(i.e. the number of non-empty areas in the Venn diagram generated by $\mathcal{F}$).

$\mathcal{F} = \{C_1, C_2, C_3\} \qquad |S_{\mathcal{F}}| = 8 = 2^3$



Let $\mathcal{C}$ family of subsets of a set X.

The *dual shatter function* $\pi^*_{\mathcal{C}} : \omega \to \omega$ of $\mathcal{C}$ is given by

$$\pi^*_{\mathcal{C}}(n) = \max\{|S_{\mathcal{F}}| : \mathcal{F} \subseteq \mathcal{C},\ |\mathcal{F}| = n\}.$$

E.g. $\pi^*_{\text{rectangles}}(3) = 8 = 2^3$.

Let $\mathcal{C}$ family of subsets of a set X.

The *dual shatter function* $\pi^*_{\mathcal{C}} : \omega \to \omega$ of $\mathcal{C}$ is given by

$$\pi^*_{\mathcal{C}}(n) = \max\{|S_{\mathcal{F}}| : \mathcal{F} \subseteq \mathcal{C},\ |\mathcal{F}| = n\}.$$

E.g. $\pi^*_{\text{rectangles}}(3) = 8 = 2^3$.

We know that $\pi^*_{\mathcal{C}}(n) \leq 2^n$ for all n.

Let $\mathcal{C}$ family of subsets of a set X.

The *dual shatter function* $\pi^*_{\mathcal{C}} : \omega \to \omega$ of $\mathcal{C}$ is given by

$$\pi^*_{\mathcal{C}}(n) = \max\{|S_{\mathcal{F}}| : \mathcal{F} \subseteq \mathcal{C},\ |\mathcal{F}| = n\}.$$

E.g. $\pi^*_{\text{rectangles}}(3) = 8 = 2^3$.

We know that $\pi^*_{\mathcal{C}}(n) \leq 2^n$ for all n.

If $\pi^*_{\mathcal{C}}(n) = O(n^d)$ for some d then $\mathcal{C}$ is a *Vapnik-Chervonenkis class (VC class)*.

Let $\mathcal{C}$ family of subsets of a set X.

The *dual shatter function* $\pi^*_{\mathcal{C}} : \omega \to \omega$ of $\mathcal{C}$ is given by

$$\pi^*_{\mathcal{C}}(n) = \max\{|S_{\mathcal{F}}| : \mathcal{F} \subseteq \mathcal{C}, |\mathcal{F}| = n\}.$$

E.g. $\pi^*_{\text{rectangles}}(3) = 8 = 2^3$.

We know that $\pi^*_{\mathcal{C}}(n) \leq 2^n$ for all n.

If $\pi^*_{\mathcal{C}}(n) = O(n^d)$ for some d then $\mathcal{C}$ is a *Vapnik-Chervonenkis class (VC class).*

The *VC-codensity* of a VC class $\mathcal{C}$ is

$$\text{vc}^*(\mathcal{C}) = \inf\{r \geq 0 : \pi^*_{\mathcal{C}}(n) = O(n^r)\}.$$

E.g. $\text{vc}^*(\text{rectangles}) = 2$.

Alon-Kleitman-Matoušek (p, q)-theorem (2004)

Let $\mathcal{C}$ be a VC class. Then, for any integers $p \geq q > \mathrm{vc}^*(\mathcal{C})$, there exists some n such that, for any subfamily $\mathcal{F} \subseteq \mathcal{C}$ with the (p, q)-property can be partitioned into at most n consistent subfamilies.

Matoušek proved a fractional Helly theorem for VC classes and pointed to the Alon-Kleitman method to extract a (p, q)-theorem.

The model-theoretic framework (some standard conventions)

Let T be a theory and $M \models T$.

We identify a formula $\varphi(x)$ with the set

$$\varphi(M) = \{a \in M^x : M \models \varphi(a)\}.$$

We identify a (partitioned) formula $\varphi(x, y)$ with the definable family of sets

$$\{\varphi(M, b) : b \in M^y\}.$$

So a formula $\varphi(x, y)$

- may be consistent, have a (p, q)-property...
- has dual shatter function π^*_φ,
- has VC-codensity $\mathrm{vc}^*(\varphi)$.

(All these are independent of M.)

The model-theoretic framework (some standard conventions)

Let T be a theory and $M \models T$.

We identify a formula $\varphi(x)$ with the set

$$\varphi(M) = \{a \in M^x : M \models \varphi(a)\}.$$

We identify a (partitioned) formula $\varphi(x, y)$ with the definable family of sets

$$\{\varphi(M, b) : b \in M^y\}.$$

So a formula $\varphi(x, y)$

- may be consistent, have a (p, q)-property...
- has dual shatter function π^*_φ,
- has VC-codensity $\mathrm{vc}^*(\varphi)$.

(All these are independent of M.)

A formula $\varphi(x, y)$ is *NIP* (not the independence property) if it is a VC class.
A theory/structure is *NIP* if every formula φ is NIP.

So the A.K.M. (p, q)-theorem applies to NIP formulas/definable families of sets.

Alon-Kleitman-Matoušek (p, q)-theorem (2004)

Let $\mathcal{C}$ be an NIP definable family of sets.
For any integers $p \geq q > \mathrm{vc}^*(\mathcal{C})$, there exists some n such that any subfamily $\mathcal{F} \subseteq \mathcal{C}$ with the (p, q)-property can be partitioned into at most n consistent subfamilies.

So the A.K.M. (p, q)-theorem applies to NIP formulas/definable families of sets.

Alon-Kleitman-Matoušek (p, q)-theorem (2004)

Let $\mathcal{C}$ be an NIP definable family of sets.
For any integers $p \geq q > \text{vc}^*(\mathcal{C})$, there exists some n such that any subfamily $\mathcal{F} \subseteq \mathcal{C}$ with the (p, q)-property can be partitioned into at most n consistent subfamilies.

In NIP model theory the A.K.M. (p, q)-theorem has been used to prove (Chernikov, Simon 2015) existence of uniform honest definitions and uniform definability of types over finite sets (UDTFS).

So the A.K.M. (p, q)-theorem applies to NIP formulas/definable families of sets.

Alon-Kleitman-Matoušek (p, q)-theorem (2004)

Let $\mathcal{C}$ be an NIP definable family of sets.
For any integers $p \geq q > \mathrm{vc}^*(\mathcal{C})$, there exists some n such that any subfamily $\mathcal{F} \subseteq \mathcal{C}$ with the (p, q)-property can be partitioned into at most n consistent subfamilies.

In NIP model theory the A.K.M. (p, q)-theorem has been used to prove (Chernikov, Simon 2015) existence of uniform honest definitions and uniform definability of types over finite sets (UDTFS).

If $\mathcal{F}$ with the (p, q)-property is definable, can the consistent subfamilies be chosen definable too?

Forking and dividing

Forking and dividing

Fix a structure M and monster extension U.
A formula $\varphi(x, b)$, $b \in U^b$, does **not** *divide over M* if the family

$$\{\varphi(x, c) : c \in U^b,\ \mathrm{tp}(c/M) = \mathrm{tp}(b/M)\}$$

has the (ω, k)-property for every k.

The A.K.M (p, q)-theorem implies: it suffices to have the (ω, k)-property for some $k > \mathrm{vc}^*(\varphi)$.

Forking and dividing

Fix a structure M and monster extension U.
A formula $\varphi(x, b)$, $b \in U^b$, does **not** *divide over* M if the family

$$\{\varphi(x, c) : c \in U^b,\ \mathrm{tp}(c/M) = \mathrm{tp}(b/M)\}$$

has the (ω, k)-property for every k.

The A.K.M (p, q)-theorem implies: it suffices to have the (ω, k)-property for some $k > \mathrm{vc}^*(\varphi)$.

Overall in NIP theories TFAE:

- $\varphi(x, b)$ does not fork over M.
- $\varphi(x, b)$ does not divide over M.
- $\{\varphi(x, c) : c \in U^b,\ \mathrm{tp}(c/M) = \mathrm{tp}(b/M)\}$ is consistent.
- $\{\varphi(x, c) : c \in U^b,\ \mathrm{tp}(c/M) = \mathrm{tp}(b/M)\}$ extends to an M-invariant type.

- $\{\varphi(x, c) : c \in U^b,\ \mathrm{tp}(c/M) = \mathrm{tp}(b/M)\}$ extends to an M-invariant type.

- $\{\varphi(x,c) : c \in U^b,\ \mathrm{tp}(c/M) = \mathrm{tp}(b/M)\}$ extends to an M-invariant type.

Definable (p,q)-conjecture (Simon 2015)

Let $\varphi(x,y)$ be an NIP formula. If $\varphi(x,b)$ does not divide over M then there exists a formula $\psi(y) \in \mathrm{tp}(b/M)$ such that $\{\varphi(x,c) : U \models \psi(c)\}$ is consistent.

- $\{\varphi(x,c) : c \in U^b,\ \mathrm{tp}(c/M) = \mathrm{tp}(b/M)\}$ extends to an M-invariant type.

Definable (p,q)-conjecture (Simon 2015)

Let $\varphi(x,y)$ be an NIP formula. If $\varphi(x,b)$ does not divide over M then there exists a formula $\psi(y) \in \mathrm{tp}(b/M)$ such that $\{\varphi(x,c) : U \models \psi(c)\}$ is consistent.

By model-theoretic compactness, plus the A.K.M. (p,q)-theorem, this is equivalent to:

Let $\mathcal{C}$ be an NIP definable family of sets with the (p,q)-property for some $p \geq q > \mathrm{vc}^*(\mathcal{C})$. Then $\mathcal{C}$ partitions into finitely many consistent definable subfamilies.

- Simon and Starchenko (2014) prove a stronger version of the conjecture for some dp-minimal theories (e.g. weakly-o-minimal, ACVF, Pres. Ar., p-adics).
- Simon (2015) states the conjecture and proves it for NIP theories with small or medium directionality (a notion which measures the number of coheirs).
- Boxall and Kestner (2018) prove the conjecture for distal theories.
- Rakotonarivo (2021) proves the conjecture for certain dense pairs of geometric distal structures.

Theorem (AG., 2023)

Let $\mathcal{C}$ be a definable family of sets with $\mathrm{vc}^*(\mathcal{C}) < 2$. If $\mathcal{C}$ has the $(\omega, 2)$-property then $\mathcal{C}$ can be parititioned into finitely many consistent definable subfamilies.

Theorem (AG., 2023)

Let $\mathcal{C}$ be a definable family of sets with $\mathrm{vc}^*(\mathcal{C}) < 2$. If $\mathcal{C}$ has the $(\omega, 2)$-property then $\mathcal{C}$ can be paritioned into finitely many consistent definable subfamilies.

This improves the A.K.M. (p, q)-theorem in the case $\mathrm{vc}^*(\mathcal{C}) < 2$ ($(\omega, 2)$-property is weaker than the $(p, 2)$-property for $p < \omega$).

Theorem (AG., 2023)

Let $\mathcal{C}$ be a definable family of sets with $\mathrm{vc}^*(\mathcal{C}) < 2$. If $\mathcal{C}$ has the $(\omega, 2)$-property then $\mathcal{C}$ can be paritioned into finitely many consistent definable subfamilies.

This improves the A.K.M. (p, q)-theorem in the case $\mathrm{vc}^*(\mathcal{C}) < 2$ ($(\omega, 2)$-property is weaker than the $(p, 2)$-property for $p < \omega$).

Theorem (Kaplan, 2024)

Let M be an NIP structure. Every definable family $\mathcal{C}$ with the (p, q)-property, for $p \geq q > \mathrm{vc}^*(\mathcal{C})$, can be partitioned into finitely many consistent definable subfamilies.

Theorem (AG., 2023)

Let $\mathcal{C}$ be a definable family of sets with $\mathrm{vc}^*(\mathcal{C}) < 2$. If $\mathcal{C}$ has the $(\omega, 2)$-property then $\mathcal{C}$ can be paritioned into finitely many consistent definable subfamilies.

This improves the A.K.M. (p, q)-theorem in the case $\mathrm{vc}^*(\mathcal{C}) < 2$ ($(\omega, 2)$-property is weaker than the $(p, 2)$-property for $p < \omega$).

Theorem (Kaplan, 2024)

Let M be an NIP structure. Every definable family $\mathcal{C}$ with the (p, q)-property, for $p \geq q > \mathrm{vc}^*(\mathcal{C})$, can be partitioned into finitely many consistent definable subfamilies.

For a formula $\varphi(x, y_1, y_2)$ and $p \geq q > \mathrm{vc}^*(\varphi)$ there is n such that, for all $c \in M^{y_2}$, if $\varphi(x, y_1, c)$ has the (p, q)-property then it can be partitioned into $\leq n$ consistent definable subfamilies.

Above $\varphi(x, y_1, c)$ is identified with the family of sets $\varphi(M, b, c)$ for $b \in M^{y_1}$.

So, what now...

- Definable (p, q)-conjecture for NIP formulas remains open.
- Stronger uniform versions remain open.
(e.g. can every definable subfamily with the (p, q)-property be partitioned into $n = n(p, q)$ consistent subfamilies.)
- In the A.K.M. (p, q)-theorem, can the (p, q)-property be relaxed to the (ω, q)-property?