O-minimal tame set-theoretic topology

Pablo Andújar Guerrero

Model and Sets Seminar

Definition

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I.e. there is a formula $\varphi(\bar{x}, \bar{y})$ such that $\mathcal{B}$ is the family of sets

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Examples

Let $\mathcal{M} = (M, <, \ldots)$ expand a dense linear order.

- Euclidean topology (τ_e) on M.

  $\mathcal{B} = \{(b_1, b_2) : b_1 < b_2\}$, $\varphi(x, y_1, y_2) =$ "$y_1 < x < y_2$".

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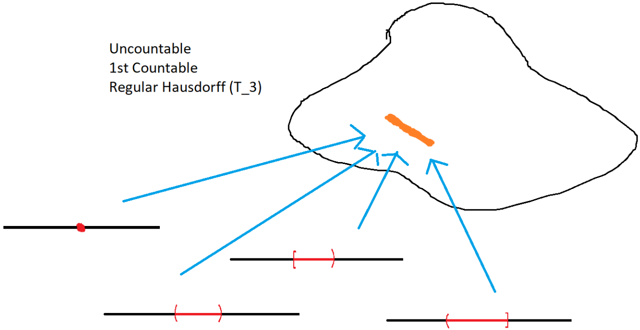
- Right half-open interval topology (τ_r) on M (Sorgenfrey line).
- Left half-open interval topology (τ_l) on M.
- Discrete topology (τ_s).

  $\mathcal{B} = \{\{b\} : b \in M\}$, $\varphi(x, y) =$ "$x = y$".

Conjecture A (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that, for every uncountable first countable regular Hausdorff topological space (X, τ), there exists a uncountable set $Y \subseteq \mathbb{R}$ and an embedding $f : (Y, \mu) \hookrightarrow (X, \tau)$, where $\mu \in \{\tau_e, \tau_r, \tau_s\}$.



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Theorem (AG, Thomas, Walsberg)

Let (X, τ) be an infinite T_1 (singletons are closed) definable topological space in an o-minimal structure $\mathcal{M}$. There exists an interval $I \subseteq M$ and a definable embedding $f : (I, \mu) \hookrightarrow (X, \tau)$, where $\mu \in \{\tau_e, \tau_r, \tau_l, \tau_s\}$.

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Let $\mathcal{B}$ be a definable basis for τ and, for $x \in X$, let $\mathcal{B}_x = \{A \in \mathcal{B} : x \in A\}$ be a definable basis of neighborhoods of x.

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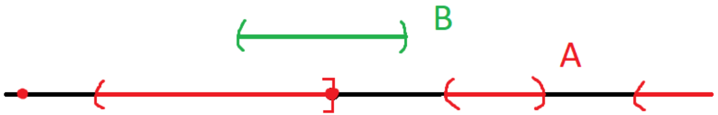

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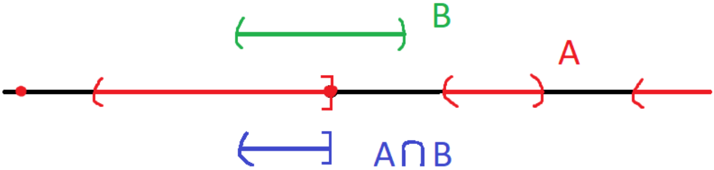

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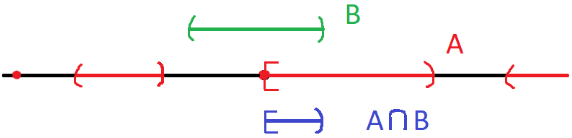

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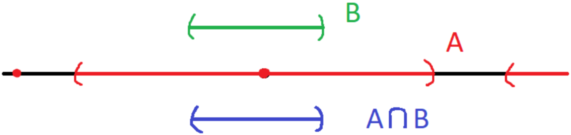

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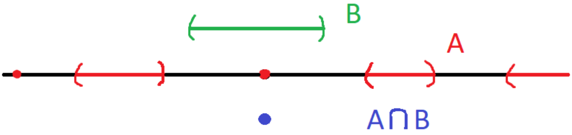
If $I \cap (J_1 \cup J_2) = \emptyset \Rightarrow \tau|_I = \tau_s$.



Given a class of topological spaces $\mathcal{C}$, a *basis* for $\mathcal{C}$ is a subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that every space in $\mathcal{C}$ contains a homeomorphic copy of a space in $\mathcal{C}_0$.

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Let us denote the condition in blue by $(\dagger)$

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$ZFC + (\dagger)$ is consistent.

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Moore (2006) proves that the first countability assumption is necessary (construction of an L-space from ZFC).

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Gruenhage (1989) proves that, under PFA, $(\dagger)$ holds for the class of cometrizable spaces (a generalization of metric spaces). Todorcevic (1989) proves it under the Open Coloring Axiom (OCA).

Farhat (2015) observed that, under PFA, $(\dagger)$ holds for monotonically normal (a class that includes both metric and linearly ordered spaces) compacta.

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Refined Conjecture A*

$ZFC + PFA \Rightarrow (\dagger)$.

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Analysis of a topological basis problem

Y. Peng & S. Todorcevic

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Abstract

We examine a basis problem for uncountable regular first countable spaces using the Proper Forcing Axiom. We introduce a notion of inner and outer topologies and show that they come quite close to characterizing the correctness of the current conjecture about this basis problem.

I have two doubts regarding all this.

These are used interchangeably in the literature.

Conjecture A (3-element Basis Conjecture, Gruenhage '86)

It is consistent with ZFC that, for every uncountable first countable T_3 topological space (X, τ), there exists a uncountable subset of the reals with the euclidean, discrete or Sorgenfrey line topology that embeds into it.

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It is consistent with ZFC that the class of uncountable first countable T_3 topological spaces has a 3-element basis given by an uncountable discrete space, an uncountable subspace of the reals (with the euclidean topology), and an uncountable subspace of the Sorgenfrey line.

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Are these equivalent?

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Theorem (Gruenhage '89)

Assume PFA. Let (X, τ) be a cometrizable space with no uncountable discrete subspace. Then either

1. (X, τ) contains a copy of an uncountable subspace of the Sorgenfrey line; or
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1. (X, τ) contains a copy of an uncountable subspace of the Sorgenfrey line; or
2. X is the continuous image of a separable metric space (i.e. cosmic).

Does being cosmic imply containing a copy of an uncountable subspace of the real line?

A related problem

Fremlin's Conjecture

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Is it consistent with ZFC that every perfect (no isolated points) Hausdorff compactum admits a continuous and at most two-to-one map onto a metric space?

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Fremlin's conjecture is equivalent, under PFA, to the basis conjecture for subspaces of perfectly normal compacta.

Question: Is there an o-minimal definable positive answer to Fremlin's Conjecture?

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Theorem (AG, Thomas, Walsberg)

Let (X, τ) be a Hausdorff definable topological space in an o-minimal structure $\mathcal{M}$, with $X \subseteq M$. There exists a finite partition $\mathcal{I}$ of X into points and intervals such that, for every interval $I \in \mathcal{I}$, the subspace topology $\tau|_I$ is one of τ_e, τ_r, τ_l or τ_s.

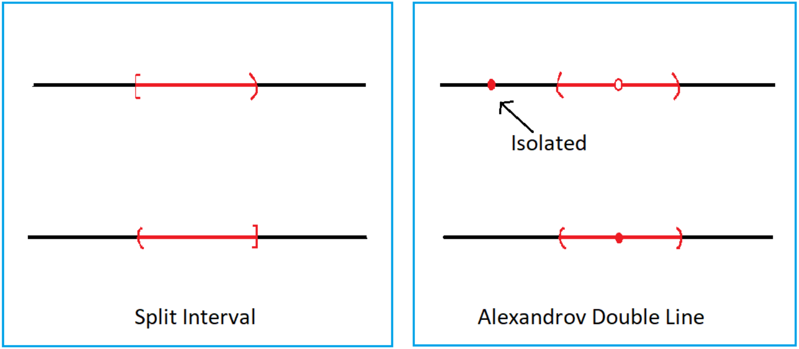
The *split interval* is the set $[0,1] \times \{0,1\}$ with the lexicographic order topology τ_{lex}.

The *Alexandrov double line* is the set $[0,1] \times \{0,1\}$ with the following topology τ_{Alex}:

- All the points in $[0,1] \times \{1\}$ are isolated.
- Basic neighborhoods of points $\langle x, 0 \rangle$ are of the form $(y,z) \times \{0,1\} \setminus \{\langle x, 1 \rangle\}$ for $y < x < z$.

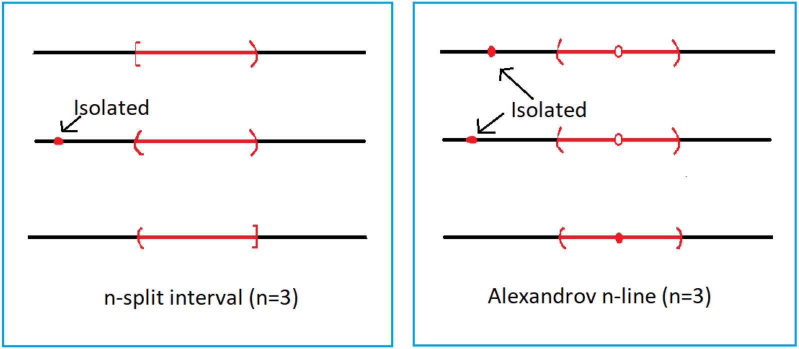
The split interval: $([0,1] \times \{0,1\}, \tau_{lex})$.

The Alexandrov double line: $([0,1] \times \{0,1\}, \tau_{Alex})$.



The n-split interval: $(\mathbb{R} \times \{0, \ldots, n-1\}, \tau_{lex})$.

The Alexandrov n-line: $(\mathbb{R} \times \{0, \ldots, n-1\}, \tau_{Alex})$.



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Theorem (AG, Thomas, Walsberg)

Let (X, τ) be a Hausdorff regular definable topological space in an o-minimal structure $\mathcal{M}$, with $X \subseteq M$. There exist disjoint definable open sets Y and Z, with $X \setminus (Y \cup Z)$ finite, and some $n < \omega$, such that

1. There space (Y, τ) embeds definably into the definable n-split interval.
2. There space (Z, τ) embeds definably into the definable Alexandrov n-line.

As a consequence we derive a definable o-minimal positive answer to Fremlin's conjecture, for 1-dimensional spaces.

Corollary

Let (X, τ) be a perfect Hausdorff regular definable topological space in an o-minimal structure $\mathcal{M}$, with $X \subseteq M$. Then (X, τ) admits a definable continuous and at most two-to-one map into (M, τ_e).

Thank you.